THE DIFFERENTIAL OF A FUNCTION AND METHODS OF ITS CALCULATION

Boboqulova Durdona Sanjar qizi
First-year student of the Mathematics Department,
Faculty of Pedagogy, Shahrisabz State Pedagogical Institute

Abstract: This article explores the concept of the differential of a function in calculus. The differential provides a linear approximation of a function's change in response to small variations in its input. The paper discusses the formal definition of differentials, their relationship with derivatives, rules of computation, and practical applications in approximation, physics, and engineering. Special attention is given to methods of calculating differentials for single-variable and multivariable functions.

Keywords: differential, derivative, linear approximation, multivariable calculus, total differential, tangent plane, error estimation.

The differential is one of the core concepts in differential calculus. While the derivative provides the rate of change of a function, the differential expresses this rate in terms of a small change in the input variable. It is an essential tool in mathematical modeling, enabling linear approximations and error analysis for functions that may not be easily evaluated directly.

The notion of differentials has evolved historically from infinitesimal calculus to rigorous mathematical analysis. In modern applications, differentials are widely used in physics (e.g., small changes in thermodynamic quantities), economics (e.g., marginal analysis), and numerical methods (e.g., estimating values of functions without exact computation).

This paper introduces the formal definition of the differential and presents methods for calculating differentials in both one and several variables.

The research methodology involves:

- Defining differentials based on derivatives.
- Deriving differential formulas for elementary functions.
- Applying rules such as linearity, chain rule, and total differential.
- Demonstrating practical use through examples and approximations.
- Extending the concept to multivariable calculus.

Definition of the Differential

Let y=f(x)y = f(x)y=f(x) be a differentiable function. The differential of fff, denoted dydydy, is defined as:

dy=f′(x) dxdy = f'(x) \, dxdy=f′(x)dx

Here:

- dxdxdx is an independent variable representing a small change in xxx,
- dydydy approximates the corresponding change in yyy,

- The derivative f′(x)f'(x)f′(x) acts as a scaling factor.

This means that for small Δx\Delta xΔx, the change in the function Δy≈dy=f′(x)⋅dx\Delta y \approx dy = f'(x) \cdot dxΔy≈dy=f′(x)⋅dx.
Example: Computing a Differential
Let f(x)=x3f(x) = x^3f(x)=x3. Then:
f′(x)=3x2⇒dy=3x2dxf'(x) = 3x^2 \quad \Rightarrow \quad dy = 3x^2 dxf′(x)=3x2⇒dy=3x2dx
For x=2x = 2x=2 and dx=0.1dx = 0.1dx=0.1, the approximate change in yyy is:
dy=3(2)2⋅0.1=1.2dy = 3(2)^2 \cdot 0.1 = 1.2dy=3(2)2⋅0.1=1.2
This gives an approximation of how much yyy changes when xxx increases from 2 to 2.1.
Differential Rules
Differentials obey the same rules as derivatives:

- Linearity:

d(af(x)+bg(x))=a df(x)+b dg(x)d(af(x) + bg(x)) = a \, df(x) + b \, dg(x)d(af(x)+bg(x))=adf(x)+bdg(x)

- Product Rule:

d(f(x)g(x))=f(x) dg(x)+g(x) df(x)d(f(x)g(x)) = f(x) \, dg(x) + g(x) \, df(x)d(f(x)g(x))=f(x)dg(x)+g(x)df(x)

- Quotient Rule:

d(f(x)g(x))=g(x) df(x)−f(x) dg(x)g(x)2d\left(\frac{f(x)}{g(x)}\right) = \frac{g(x) \, df(x) - f(x) \, dg(x)}{g(x)^2}d(g(x)f(x))=g(x)2g(x)df(x)−f(x)dg(x)

- Chain Rule:
  If y=f(u),u=g(x)y = f(u), u = g(x)y=f(u),u=g(x), then:

dy=dfdu⋅dudx⋅dxdy = \frac{df}{du} \cdot \frac{du}{dx} \cdot dxdy=dudf⋅dxdu⋅dx
Total Differential for Multivariable Functions
Let z=f(x,y)z = f(x, y)z=f(x,y) be a differentiable function of two variables. Then the total differential is:
dz=∂f∂x dx+∂f∂y dydz = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dydz=∂x∂fdx+∂y∂fdy
This formula provides a linear approximation to the change in zzz due to small changes in both xxx and yyy.
Example:
Let f(x,y)=x2y+y3f(x, y) = x^2y + y^3f(x,y)=x2y+y3
Then:
dz=∂f∂xdx+∂f∂ydy=2xy dx+(x2+3y2) dydz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 2xy \, dx + (x^2 + 3y^2) \, dydz=∂x∂fdx+∂y∂f dy=2xydx+(x2+3y2)dy
Differentials are used as powerful tools in both theory and practice:

- Approximations: Differentials can be used to estimate function values near a known point.
- Error Estimation: In measurements, differentials help quantify how small changes in input affect the output.
- Physics: In thermodynamics and mechanics, differentials represent infinitesimal quantities (e.g., dQdQdQ, dWdWdW, dUdUdU).
- Multivariable Analysis: The total differential leads to the tangent plane approximation, Jacobian matrices, and differential forms in advanced calculus.

In numerical methods, differentials provide the foundation for Newton's method, Euler's method, and finite difference approximations.

Despite being small, differentials represent the bridge between discrete changes and continuous behavior — a cornerstone of calculus.

The differential of a function serves as a key concept for understanding and estimating changes in mathematical functions. It translates the derivative into a linear approximation of function change, useful in a wide range of scientific and engineering contexts.

This paper reviewed the definition of differentials, rules of calculation, and extended the idea to multivariable functions. The differential is not only a theoretical concept but also a practical tool in error analysis, approximation, and optimization.

Future studies may explore the role of differentials in differential geometry, differential equations, and machine learning optimization algorithms.

References

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