GROUND STATES OF TRANSLATION-INVARIANT TYPE FOR THE FOUR-STATE CHUI-WEEKS MODEL ON THE CAYLEY TREE OF ORDER TWO

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Abstract: This paper investigates the four-state Chui-Weeks model defined on a Cayley tree of order two. A complete description of all translation-invariant ground states for this model is provided.

Key words: Cayley tree, Chui-Weeks model, ground state, translation invariance.

In this study, we analyze the four-state Chui-Weeks model on the Cayley tree of order two for the first time. All possible translation-invariant ground states are characterized in detail. Earlier, the three-state version of the Chui-Weeks model on Cayley trees was examined in [1–3].

The Cayley tree Γ^k (see, e.g., [4]) of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, from each vertex of which exactly $k+1$ edges issue. Let $\Gamma^k = (V, L, i)$, where V is the set of vertices of Γ^k , L is the set of edges of Γ^k and i is the incidence function associating each edge $l \epsilon L$ with its endpoints $x, y \epsilon V$. If $i(l) = \{x, y\}$, then x and y are called nearest neighboring vertices, and we write $l = \langle x, y \rangle$.

We consider the model where the spin takes values in the set $\Phi = \{0, 1, 2, 3\}$. For $A \subseteq V$ a spin configuration σ_A on A is defined as a function $x \in A \to \sigma_A(x) \in \Phi$; the set of all configurations coincides with $\Omega_A = \Phi^A$. Denote $\Omega = \Omega_V$ and $\sigma = \sigma_V$.

Definition 1. A configuration that is invariant with respect to all shifts is called translation-invariant.

The Chui-Weeks model (see [1]) is defined by the following Hamiltonian

$$H(\sigma) = J \sum_{\langle x,y \rangle \epsilon L} |\sigma(x) - \sigma(y)| + \alpha \sum_{x \epsilon V} \delta_{\sigma(x),0}, \qquad (1)$$

where $J, \alpha \epsilon R, \alpha$ is an external field and $\sigma \epsilon \Omega$.

Remark. It should be noted that model (1) reduces to the SOS model when the parameter $\alpha = 0$ (see, for example, [4]).

Let M be the set of all unit balls with vertices in V and $S_1(x)$ be the set of all nearest neighboring vertices of $x \epsilon V$.

We call the restriction of a configuration to the ball $b \epsilon M$ a bounded configuration σ_b. The energy of configuration σ_b on b is defined by the formula

$$U(\sigma_b) = \frac{J}{2} \sum_{x \epsilon S_1(c_b)} |\sigma(x) - \sigma(c_b)| + \frac{\alpha}{k+2} \sum_{x \epsilon b} \delta_{\sigma(x),0}.$$

where $J = (J, \alpha) \epsilon R^2$ and c_b is the center of the unit ball b.

The Hamiltonian (1) can be written as

$$H(\sigma) = \sum_{b \epsilon M} U(\sigma_b).$$

Now we study translation-invariant ground states for the four-state Chui-Weeks model on the Cayley tree of order two.

We have the following lemma.

Lemma. Let $k = 2$. Then for each configuration φ_b, we have the following

$$U(\varphi_b) \in \{U_i : i = 1,2, \dots, 29\},$$

where

$$U_1 = 0;\ U_2 = \alpha;\ U_3 = \frac{J}{2};\ U_4 = \frac{J}{2} + \frac{\alpha}{4};\ U_5 = \frac{J}{2} + \frac{3\alpha}{4};U_6 = J;\ U_7 = J + \frac{\alpha}{4};$$
$$U_8 = J + \frac{\alpha}{2};\ U_9 = J + \frac{3\alpha}{4};\ U_{10} = \frac{3J}{2};\ U_{11} = \frac{3J}{2} + \frac{\alpha}{4};\ U_{12} = \frac{3J}{2} + \frac{\alpha}{2};$$
$$U_{13} = \frac{3J}{2} + \frac{3\alpha}{4};\ U_{14} = 2J;\ U_{15} = 2J + \frac{\alpha}{4};\ U_{16} = 2J + \frac{\alpha}{2};\ U_{17} = \frac{5J}{2};$$
$$U_{18} = \frac{5J}{2} + \frac{\alpha}{4};\ U_{19} = \frac{5J}{2} + \frac{\alpha}{2};\ U_{20} = 3J;\ U_{21} = 3J + \frac{\alpha}{4};\ U_{22} = 3J + \frac{\alpha}{2};$$
$$U_{23} = 3J + \frac{3\alpha}{4};\ U_{24} = \frac{7J}{2} + \frac{\alpha}{4};U_{25} = \frac{7J}{2} + \frac{\alpha}{2};\ U_{26} = 4J + \frac{\alpha}{4};\ U_{27} = 4J + \frac{\alpha}{2};$$

$$U_{28} = \frac{9J}{2} + \frac{\alpha}{4};\ U_{28} = \frac{9J}{2} + \frac{3\alpha}{4}.$$

Definition 2. A configuration φ is called a ground state for the Hamiltonian (1), if

$$U(\varphi_b) = min\{U_i: i = 1,2, \ldots ,29\}$$

for any $b \in M$.

We denote

$$A_\xi = \left\{(J,\alpha) \in R^2: U_\xi = min\{U_i: i = 1,2, \ldots ,29\}\right\}.$$

Calculations show that:

$$A_1 = \{(J,\alpha) \in R^2: J \geq 0, \alpha \geq 0\};$$
$$A_2 = \{(J,\alpha) \in R^2: 18J \geq \alpha, \alpha \leq 0\};$$
$$A_3 = A_6 = A_{10} = A_{14} = A_{17} = \{(J,\alpha) \in R^2: J = 0, \alpha \geq 0\};$$
$$A_4 = A_5 = A_7 = A_8 = A_9 = A_{11} = A_{12} = A_{13} = A_{15} =$$
$$A_{16} = A_{18} = A_{19} = A_{21} = A_{22} = \cdots =$$
$$= A_{27} = \{(J,\alpha) \in R^2: J = 0, \alpha = 0\};$$
$$A_{20} = \{(J,\alpha) \in R^2: -\alpha \leq 6J \leq 0\};$$
$$A_{28} = \{(J,\alpha) \in R^2: 0 \leq \alpha \leq -6J\};$$
$$A_{29} = \{(J,\alpha) \in R^2: 18J \leq \alpha \leq 0\}$$

and $\bigcup_{i=1}^{29} A_i = R^2$.

The next theorem gives a full description of the translation-invariant ground states of the four-state Chui-Weeks model on the Cayley tree of order two.

Theorem. Let $k = 2$. Then, for the four-state Chui-Weeks model, the following assertions hold:

(i) The configurations $\sigma(x) = i, i \in \{1,2,3\}$, for for all $x \in V$ are translation-invariant ground states iff $(J,\alpha) \in A_1$;

(ii) The configuration $\sigma(x) = 0$ for all $x \in V$ is the translation-invariant ground state iff $(J,\alpha) \in A_2$;

(iii) If $(J,\alpha) \in R^2 \setminus (A_1 \cup A_2)$, then there is no translation-invariant ground state.

Proof. Let $k=2$.

(i) We consider the configuration $\sigma(x)=i, i\in\{1,2,3\}$, for all $x\in V$. For any $b\in M$ by Lemma we have $U(\sigma_b)=U_1=0.$ Thus the configuration $\sigma(x)=i, i\in\{1,2,3\}$, for all $x\in V$ is ground state iff $(J,\alpha)\in A_1$;

(ii) We consider the configuration $\sigma(x)=0$ for all $x\in V$. For any $b\in M$ by Lemma we have $U(\sigma_b)=U_2=\alpha.$ Thus the configuration $\sigma(x)=0$ for all $x\in V$ is ground state iff $(J,\alpha)\in A_2$;

(iii) It is obvious. The theorem has been proved.

LITERATURE

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