TRANSLATION-INVARIANT GROUND STATES FOR THE CHUI–WEEKS MODEL WITH FOUR SPIN STATES ON A CAYLEY TREE OF ORDER TWO

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Abstract: In this work, the four-state Chui-Weeks model on the second-order Cayley tree is considered. For this model, all translation-invariant ground states are described.

Key words: Cayley tree, Chui-Weeks model, ground state, translation-invariant ground state.

In the present work, the four-state Chui-Weeks model is considered on a second-order Cayley tree for the first time and for this model, all translation-invariant ground states are described. The three-state Chui-Weeks model on Cayley trees was studied in [1-3].

The Cayley tree Γ^k (see, e.g., [4]) of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, from each vertex of which exactly $k+1$ edges issue. Let $\Gamma^k = (V, L, i)$, where V is the set of vertices of Γ^k , L is the set of edges of Γ^k and

i is the incidence function associating each edge $l \epsilon L$ with its endpoints $x, y \epsilon V$. If $i(l) = \{x, y\}$, then x and y are called nearest neighboring vertices, and we write $l = \langle x, y \rangle$.

We consider the model where the spin takes values in the set $\Phi = \{0, 1, 2, 3\}$. For $A \subseteq V$ a spin configuration σ_A on A is defined as a function $x \in A \to \sigma_A(x) \in \Phi$; the set of all configurations coincides with $\Omega_A = \Phi^A$. Denote $\Omega = \Omega_V$ and $\sigma = \sigma_V$.

Definition 1. A configuration that is invariant with respect to all shifts is called translation-invariant.

The Chui-Weeks model (see [1]) is defined by the following Hamiltonian

$$H(\sigma) = J \sum_{\langle x,y\rangle \epsilon L} |\sigma(x) - \sigma(y)| + \alpha \sum_{x \epsilon V} \delta_{\sigma(x),0}, \qquad (1)$$

where $J, \alpha \epsilon R, \alpha$ is an external field and $\sigma \epsilon \Omega$.

Remark 1. Recall that model (1) coincides with the SOS model under the condition $\alpha = 0$ (see, e.g., [4]).

Let M be the set of all unit balls with vertices in V and $S_1(x)$ be the set of all nearest neighboring vertices of $x \epsilon V$.

We call the restriction of a configuration to the ball $b \epsilon M$ a bounded configuration σ_b. The energy of configuration σ_b on b is defined by the formula

$$U(\sigma_b) = \frac{J}{2} \sum_{x \epsilon S_1(c_b)} |\sigma(x) - \sigma(c_b)| + \frac{\alpha}{k+2} \sum_{x \epsilon b} \delta_{\sigma(x),0}.$$

where $J = (J, \alpha) \epsilon R^2$ and c_b is the center of the unit ball b.

The Hamiltonian (1) can be written as

$$H(\sigma) = \sum_{b \epsilon M} U(\sigma_b).$$

Now we study translation-invariant ground states for the four-state Chui-Weeks model on the Cayley tree of order two.

We have the following lemma.

Lemma 1. Let $k = 2$. Then for each configuration φ_b, we have the following

$$U(\varphi_b) \in \{U_i : i = 1,2, \ldots, 29\},$$

where

$U_1=0;\ U_2=\alpha;\ U_3=\frac{J}{2};\ U_4=\frac{J}{2}+\frac{\alpha}{4};\ U_5=\frac{J}{2}+\frac{3\alpha}{4};U_6=J;\ U_7=J+\frac{\alpha}{4};$

$$U_8=J+\frac{\alpha}{2};\ U_9=J+\frac{3\alpha}{4};\ U_{10}=\frac{3J}{2};\ U_{11}=\frac{3J}{2}+\frac{\alpha}{4};\ U_{12}=\frac{3J}{2}+\frac{\alpha}{2};$$

$$U_{13}=\frac{3J}{2}+\frac{3\alpha}{4};\ U_{14}=2J;\ U_{15}=2J+\frac{\alpha}{4};\ U_{16}=2J+\frac{\alpha}{2};\ U_{17}=\frac{5J}{2};$$

$$U_{18}=\frac{5J}{2}+\frac{\alpha}{4};\ U_{19}=\frac{5J}{2}+\frac{\alpha}{2};\ U_{20}=3J;\ U_{21}=3J+\frac{\alpha}{4};\ U_{22}=3J+\frac{\alpha}{2};$$

$$U_{23}=3J+\frac{3\alpha}{4};\ U_{24}=\frac{7J}{2}+\frac{\alpha}{4};U_{25}=\frac{7J}{2}+\frac{\alpha}{2};\ U_{26}=4J+\frac{\alpha}{4};\ U_{27}=4J+\frac{\alpha}{2};$$

$$U_{28}=\frac{9J}{2}+\frac{\alpha}{4};\ U_{28}=\frac{9J}{2}+\frac{3\alpha}{4}.$$

Definition 2. A configuration φ is called a ground state for the Hamiltonian (1), if

$$U(\varphi_b)=min\{U_i: i=1,2,\dots,29\}$$

for any $b\in M$.

We denote

$$A_\xi=\left\{(J,\alpha)\in R^2: U_\xi=min\{U_i: i=1,2,\dots,29\}\right\}.$$

Calculations show that:

$$A_1=\{(J,\alpha)\in R^2: J\geq 0,\alpha\geq 0\}; A_2=\{(J,\alpha)\in R^2: 18J\geq\alpha,\alpha\leq 0\};$$

$$A_3=A_6=A_{10}=A_{14}=A_{17}=\{(J,\alpha)\in R^2: J=0,\alpha\geq 0\};$$

$A_4=A_5=A_7=A_8=A_9=A_{11}=A_{12}=A_{13}=A_{15}=A_{16}=A_{18}=A_{19}=A_{21}=A_{22}=\ldots=A_{27}=\{(J,\alpha)\in R^2: J=0,\alpha=0\};$

$$A_{20}=\{(J,\alpha)\in R^2: -\alpha\leq 6J\leq 0\};\ A_{28}=\{(J,\alpha)\in R^2: 0\leq\alpha\leq -6J\};$$

$$A_{29}=\{(J,\alpha)\in R^2: 18J\leq\alpha\leq 0\};$$

and $\bigcup_{i=1}^{29}A_i=R^2$.

The following theorem describes all translation-invariant ground states for the four-state Chui-Weeks model on the Cayley tree of order two.

Theorem 1. Let $k = 2$. Then, for the four-state Chui-Weeks model, the following assertions hold:

(i) The configurations $\sigma(x) = i, i \in \{1,2,3\}$, for for all $x \in V$ are translation-invariant ground states iff $(J, \alpha) \in A_1$;

(ii) The configuration $\sigma(x) = 0$ for all $x \in V$ is the translation-invariant ground state iff $(J, \alpha) \in A_2$;

(iii) If $(J, \alpha) \in R^2 \setminus (A_1 \cup A_2)$, then there is no translation-invariant ground state.

Proof. Let $k = 2$.

(i) We consider the configuration $\sigma(x) = i, i \in \{1,2,3\}$, for all $x \in V$. For any $b \in M$ by Lemma we have $U(\sigma_b) = U_1 = 0$. Thus the configuration $\sigma(x) = i, i \in \{1,2,3\}$, for all $x \in V$ is ground state iff $(J, \alpha) \in A_1$;

(ii) We consider the configuration $\sigma(x) = 0$ for all $x \in V$. For any $b \in M$ by Lemma we have $U(\sigma_b) = U_2 = \alpha$. Thus the configuration $\sigma(x) = 0$ for all $x \in V$ is ground state iff $(J, \alpha) \in A_2$;

(iii) It is obvious. The theorem is proved.

LITERATURE

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